Science Supplementary Material
SM358 The Quantum World

SM358

Exercises on Background Maths

This set of exercises develops some key mathematical skills. It will be a great advantage in studying The Quantum World if you can carry out calculations like these accurately and fluently. The skills in Exercises 1.1 – 4.2 are all relevant for Book 1. The skills in Exercises 5.1 – 5.3 are needed for Book 2 and those in Exercise 4.3 are needed for Book 3.

To gain maximum benefit, it is important to attempt each question thoroughly before looking up the solution. If you need more help with the techniques used in a particular answer, have a look on the course website to see if a screencast solution is available.

Topic 1 — Algebra and vectors

Exercise 1.1 Given that

$$A + B = \alpha C$$

$$A - B = \beta C,$$

find B/A and C/A in terms of α and β.

Exercise 1.2 Solve the equation

$$\left(\frac{x+1}{x-1}\right)^{1/2} = \frac{a+b}{a-b},$$

where $a > b > 0$, expressing x as a simple function of a and b.

Exercise 1.3 Given two vectors, $\mathbf{a}$ and $\mathbf{b}$, each of unit magnitude with an angle of $\pi/4$ radians between their directions, evaluate (a) $\mathbf{u} \cdot \mathbf{v}$ and (b) $|\mathbf{u} \times \mathbf{v}|$, where $\mathbf{u} = \mathbf{a} + \mathbf{b}$ and $\mathbf{v} = \mathbf{a} - \mathbf{b}$.

Topic 2 — Complex numbers

Exercise 2.1 Show that

$$\left|\frac{\mathrm{i}k + \alpha}{\mathrm{i}k - \alpha}\right|^2 = 1$$

where k and α are real.

Exercise 2.2 Solve $\mathrm{e}^{\mathrm{i}x} = -1$ for x.

Exercise 2.3 Given that $z = A\mathrm{e}^{\mathrm{i}\theta} + B\mathrm{e}^{-\mathrm{i}\theta}$, where θ, A and B are real, find $|z|^2$ in terms of A, B and $\cos\theta$.

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2.1 PT/T

Topic 3 — Differentiation

Exercise 3.1 Evaluate: (a) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\mathrm{e}^{-ax^2}\right)$ (b) $\dfrac{1}{r^2}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r^2\dfrac{\mathrm{d}}{\mathrm{d}r}\mathrm{e}^{-r/a}\right)$, where a is a constant.

Exercise 3.2 Use calculus to find the minimum value of

$$f(x) = \frac{A}{x^2} + Bx^2$$

where x is a real variable and A and B are positive constants.

Exercise 3.3 Given a function $f(x, y, z) = \sin(ax)\sin(by)\sin(cz)$, where a, b and c are constants, find

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z).$$

Hence show that $f(x, y, z)$ is an eigenfunction of the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, and find the corresponding eigenvalue.

Topic 4 — Integration

Exercise 4.1 Given that

$$\int_0^\pi u^2 \sin^2 u \,\mathrm{d}u = \frac{\pi^3}{6} - \frac{\pi}{4} \quad \text{and} \quad \int_{-\infty}^{\infty} u^2 \mathrm{e}^{-u^2}\,\mathrm{d}u = \frac{\sqrt{\pi}}{2},$$

evaluate the integrals

(a) $I = \displaystyle\int_0^{L/3} x^2 \sin^2\left(\frac{3\pi x}{L}\right)\mathrm{d}x$ (b) $J = \displaystyle\int_{-\infty}^{\infty} \frac{x^2}{b^2}\,\mathrm{e}^{-x^2/a^2}\,\mathrm{d}x$, where a and b are positive constants.

Exercise 4.2 In this question, $f(x)$ is an even function and $g(x)$ is an odd function, so $f(-x) = f(x)$ and $g(-x) = -g(x)$ for all x. Which of the following integrals can immediately be recognized as being equal to zero?

$$I_1 = \int_0^a g(x)\,\mathrm{d}x \qquad I_5 = \int_{-a}^a f(x)\,\frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x$$

$$I_2 = \int_{-a}^a f(x)\,g(x)\,\mathrm{d}x \qquad I_6 = \int_{-a}^a g(x)\,\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x$$

$$I_3 = \int_{-a}^a f^2(x)\,g(x)\,\mathrm{d}x \qquad I_7 = \int_{-a}^a f(x)\,\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x$$

$$I_4 = \int_{-a}^a g^2(x)\,f(x)\,\mathrm{d}x \qquad I_8 = \int_{-a}^a g(x)\,\frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x,$$

where a is a positive constant.

Exercise 4.3 Find the volume integral of $f(r, \theta, \phi) = r\sin\theta$, where r, θ and ϕ are spherical coordinates, over the interior of a sphere of radius R centred on the origin. You may use the fact that

$$\int_0^\pi \sin^2\theta\,\mathrm{d}\theta = \frac{\pi}{2}.$$

Topic 5 — Matrices

Exercise 5.1 Evaluate

$$\begin{bmatrix} 1 & -\mathrm{i} \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}.$$

Exercise 5.2 Given the matrices

$$\mathrm{A} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad \mathrm{B} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix},$$

evaluate $\mathrm{AB} - \mathrm{BA}$. Do the matrices A and B commute with one another?

Exercise 5.3 The matrix

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$$

has eigenvalues 9 and -1. Find normalized eigenvectors corresponding to each of these eigenvalues, and show that these two eigenvectors are orthogonal.

Solutions

Topic 1 — Algebra and vectors

Solution 1.1 Given that

$$A + B = \alpha C$$

$$A - B = \beta C,$$

we multiply the first equation by β and the second equation by α to obtain

$$\beta A + \beta B = \alpha\beta C = \alpha A - \alpha B.$$

So, eliminating C,

$$(\alpha + \beta)B = (\alpha - \beta)A$$

and

$$\frac{B}{A} = \frac{\alpha - \beta}{\alpha + \beta}.$$

Also, adding the two given equations gives $2A = (\alpha + \beta)C$, so that

$$\frac{C}{A} = \frac{2}{\alpha + \beta}.$$

Comment To find B/A in terms of α and β, we must eliminate C. The first steps in the calculation do this, giving an equation involving A, B, α and β which can be rearranged to find B/A. Similarly, to obtain an expression for C/A, we eliminate B.

Solution 1.2 Squaring both sides of the equation gives

$$\frac{x + 1}{x - 1} = \frac{(a + b)^2}{(a - b)^2}.$$

So

$$(x + 1)(a - b)^2 = (x - 1)(a + b)^2.$$

Collecting terms in x then gives

$$x\left[(a + b)^2 - (a - b)^2\right] = (a - b)^2 + (a + b)^2.$$

Using

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a - b)^2 = a^2 - 2ab + b^2,$$

we then have

$$4abx = 2a^2 + 2b^2$$

so

$$x = \frac{a^2 + b^2}{2ab}.$$

Comment The quantity we want (x) is buried inside other functions. The first steps in the calculation are designed to unwrap x, so that it appears by itself. Squaring both sides of the initial equation is a good start. We then multiply by $(a - b)^2$ and $(a + b)^2$ and collect terms in x.

Solution 1.3 (a) For the scalar product,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{b}\\ &= |a|^2-|b|^2=0\end{aligned}$$

because $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ and both $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude, 1.

(b) For the vector product,

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\times\mathbf{a}+\mathbf{b}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{b}\\ &= 2\,\mathbf{b}\times\mathbf{a}\end{aligned}$$

because the vector product of any vector with itself is zero and $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$. Hence,

$$|\mathbf{u}\times\mathbf{v}| = 2ab\sin(\pi/4) = 2\times1\times1\times\frac{1}{\sqrt{2}}=\sqrt{2}.$$

Comments (1) We can multiply out brackets involving scalar and vector products in the usual way. In the case of vector products, we must be careful to preserve the ordering of the vectors: $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ but $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.

(2) The result for the scalar product does *not* rely on the value of the angle between $\mathbf{a}$ and $\mathbf{b}$. In fact this result proves a theorem in geometry: the diagonals of a rhombus (a sheared square) are perpendicular to one another.

Topic 2 — Complex numbers

Solution 2.1 We have

$$\begin{aligned}\left|\frac{\mathrm{i}k+\alpha}{\mathrm{i}k-\alpha}\right|^2 &= \left(\frac{\mathrm{i}k+\alpha}{\mathrm{i}k-\alpha}\right)^*\left(\frac{\mathrm{i}k+\alpha}{\mathrm{i}k-\alpha}\right)\\ &= \left(\frac{-\mathrm{i}k+\alpha}{-\mathrm{i}k-\alpha}\right)\left(\frac{\mathrm{i}k+\alpha}{\mathrm{i}k-\alpha}\right).\end{aligned}$$

Swapping denominators, this gives

$$\begin{aligned}\left|\frac{\mathrm{i}k+\alpha}{\mathrm{i}k-\alpha}\right|^2 &= \left(\frac{-\mathrm{i}k+\alpha}{\mathrm{i}k-\alpha}\right)\left(\frac{\mathrm{i}k+\alpha}{-\mathrm{i}k-\alpha}\right)\\ &= (-1)\times(-1)=1.\end{aligned}$$

Comment To find $|z|^2$, we use the fact the $|z|^2=z^*z$, where the complex conjugate z^* is found by making the replacement $\mathrm{i}\to-\mathrm{i}$ throughout z.

Solution 2.2 Any complex number can be written as $r\mathrm{e}^{\mathrm{i}\theta}$. In particular, we have $\mathrm{e}^{\mathrm{i}\pi}=-1$, and more generally $\mathrm{e}^{\mathrm{i}(\pi+2n\pi)}=-1$, where n is any integer, so the solutions for x are $x=(2n+1)\pi$, where n is an integer.

Alternatively, write

$$\mathrm{e}^{\mathrm{i}x}=\cos x+\mathrm{i}\sin x,$$

and require that $\cos x=-1$ and $\sin x=0$. The solutions to the second equation are $x=m\pi$, where m is an integer, but only odd multiples of π satisfy the first equation, so we again conclude that $x=(2n+1)\pi$, where n is an integer.

Comment In pictorial terms, $\mathrm{e}^{\mathrm{i}\theta}$ represents a complex number, obtained by an anticlockwise rotation of θ radians in the complex plane starting from 1. Clearly, we reach the point -1 in the complex plane if and only if $\theta=(2n+1)\pi$, where n is an integer (positive, negative or zero).

Solution 2.3 With θ, A and B real, we have

$$\begin{aligned}
|z|^2 &= z^* z \\
&= \left(A\mathrm{e}^{\mathrm{i}\theta} + B\mathrm{e}^{-\mathrm{i}\theta}\right)^* \left(A\mathrm{e}^{\mathrm{i}\theta} + B\mathrm{e}^{-\mathrm{i}\theta}\right) \\
&= \left(A\mathrm{e}^{-\mathrm{i}\theta} + B\mathrm{e}^{\mathrm{i}\theta}\right)\left(A\mathrm{e}^{\mathrm{i}\theta} + B\mathrm{e}^{-\mathrm{i}\theta}\right) \\
&= A^2 + B^2 + AB\left[\mathrm{e}^{-2\mathrm{i}\theta} + \mathrm{e}^{2\mathrm{i}\theta}\right] \\
&= A^2 + B^2 + 2AB\cos(2\theta).
\end{aligned}$$

Using

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad \text{and} \quad \cos^2\theta + \sin^2\theta = 1,$$

we have $\cos(2\theta) = 2\cos^2\theta - 1$, so

$$\begin{aligned}
|z|^2 &= A^2 + B^2 + 2AB(2\cos^2\theta - 1) \\
&= A^2 + B^2 - 2AB + 4AB\cos^2\theta \\
&= (A-B)^2 + 4AB\cos^2\theta.
\end{aligned}$$

Comments (1) We use the fact the $|z|^2 = z^* z$ and have taken $A^* = A$ and $B^* = B$ because A and B are real. After multiplying out the brackets, the complex exponentials are combined in the usual way:

$$\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(\theta-\theta)} = \mathrm{e}^0 = 1$$

and

$$\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{2\mathrm{i}\theta} \quad \text{and} \quad \mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{-2\mathrm{i}\theta}.$$

(2) Our answer can be checked by looking at a simple case. Setting $\theta = 0$ in the final answer gives $|z|^2 = (A-B)^2 + 4AB = (A+B)^2$, which is clearly correct in this special case.

Topic 3 — Differentiation

Solution 3.1 (a) We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-ax^2}\right) = \mathrm{e}^{-ax^2} \times \frac{\mathrm{d}\left(-ax^2\right)}{\mathrm{d}x} = -2ax\,\mathrm{e}^{-ax^2}.$$

Differentiating again,

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\mathrm{e}^{-ax^2}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-2ax\,\mathrm{e}^{-ax^2}\right) \\
&= -2a\mathrm{e}^{-ax^2} + (-2ax)^2\mathrm{e}^{-ax^2} \\
&= \left(4a^2x^2 - 2a\right)\mathrm{e}^{-ax^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{e}^{-r/a}\right) &= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{r^2}{a}\mathrm{e}^{-r/a}\right) \\
&= \frac{1}{r^2}\left(-\frac{2r}{a} + \frac{r^2}{a^2}\right)\mathrm{e}^{-r/a} \\
&= \left(\frac{1}{a^2} - \frac{2}{ar}\right)\mathrm{e}^{-r/a}.
\end{aligned}$$

Comments (1) In calculating these derivatives we have used the chain rule. Taking the derivative of any exponential function delivers the same exponential function back again, but we must remember to multiply by the derivative of the contents of the exponential function.

(2) The derivatives evaluated here are used in quantum mechanics – the first in the context of a harmonic oscillator (Book 1 Chapter 5) and the second in the context of a hydrogen atom (Book 3 Chapter 2).

Solution 3.2 Differentiating with respect to x, and setting the result equal to zero gives

$$0 = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{A}{x^2} + Bx^2\right) = -\frac{2A}{x^3} + 2Bx,$$

so $x^4 = A/B$.

The second derivative is

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{2A}{x^3} + 2Bx\right) = \frac{6A}{x^4} + 2B,$$

and this is equal to $8B > 0$ at the stationary point, so the stationary value is a minimum.

Substituting $x^2 = \sqrt{A/B}$ back into expression for $f(x)$, we conclude that the minimum value of $f(x)$ is

$$f_{\text{min}} = \left(A\sqrt{\frac{B}{A}} + B\sqrt{\frac{A}{B}}\right) = 2\sqrt{AB}.$$

Comments (1) To find the value of x that minimizes a function $f(x)$, we form the derivative $\mathrm{d}f/\mathrm{d}x$ and set this equal to zero.

(2) To show that we have found a minimum (rather than a maximum or a point of inflection) we form the second derivative $\mathrm{d}^2 f/\mathrm{d}x^2$ and confirm that this is positive at the stationary point. Alternatively, we can sketch a graph of $f(x)$, which in this case will indicate that $f(x)$ has a minimum but no maxima or points of inflection.

Solution 3.3 We have

$$\frac{\partial}{\partial x}\sin(ax) = a\cos(ax)$$

$$\frac{\partial^2}{\partial x^2}\sin(ax) = \frac{\partial}{\partial x}\left(a\cos(ax)\right) = -a^2\sin(ax),$$

so

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} f(x,y,z) &= \frac{\partial^2}{\partial x^2}\sin(ax)\sin(by)\sin(cz) \\
&= \left[\frac{\partial^2}{\partial x^2}\sin(ax)\right]\sin(by)\sin(cz) \\
&= -a^2\sin(ax)\sin(by)\sin(cz) \\
&= -a^2 f(x,y,z).
\end{aligned}$$

Similar results apply for the partial differentiations with respect to y and z:

$$\frac{\partial^2}{\partial y^2} f(x,y,z) = -b^2 f(x,y,z) \quad \text{and} \quad \frac{\partial^2}{\partial z^2} f(x,y,z) = -c^2 f(x,y,z).$$

We therefore conclude that

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x,y,z) &= \frac{\partial^2}{\partial x^2} f(x,y,z) + \frac{\partial^2}{\partial y^2} f(x,y,z) + \frac{\partial^2}{\partial z^2} f(x,y,z) \\
&= -a^2 f(x,y,z) - b^2 f(x,y,z) - c^2 f(x,y,z) \\
&= -(a^2 + b^2 + c^2) f(x,y,z).
\end{aligned}$$

This shows that $f(x,y,z)$ is an eigenfunction of $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ with eigenvalue $-(a^2 + b^2 + c^2)$.

Comments (1) When $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ acts on a function, the result splits into a sum of three second-order partial derivatives. When partially differentiating with respect to x, all other variables (including y and z) are treated as constants, and similar remarks apply to the partial differentiations with respect to y and z.

(2) The quantity evaluated here is used in quantum mechanics, in the context of a particle in a three-dimensional box (Book 1 Chapter 3).

Topic 4 — Integration

Solution 4.1 (a) In the integral

$$I = \int_0^{L/3} x^2 \sin^2\left(\frac{3\pi x}{L}\right) \mathrm{d}x,$$

we make the substitution $u = 3\pi x/L$.

Then $u = 3\pi x/L$, so $x = (L/3\pi)\,u$ and $\mathrm{d}x = (L/3\pi)\,\mathrm{d}u$. Also, $x = 0$ corresponds to $u = 0$ and $x = L/3$ corresponds to $u = \pi$, so

$$I = \left(\frac{L}{3\pi}\right)^3 \int_0^{\pi} u^2 \sin^2 u \,\mathrm{d}u = \left(\frac{L}{3\pi}\right)^3 \left(\frac{\pi^3}{6} - \frac{\pi}{4}\right).$$

(b) In the integral

$$J = \int_{-\infty}^{\infty} \frac{x^2}{b^2}\,\mathrm{e}^{-x^2/a^2}\,\mathrm{d}x,$$

we can immediately take the constant $1/b^2$ outside the integral to give

$$J = \frac{1}{b^2}\int_{-\infty}^{\infty} x^2\,\mathrm{e}^{-x^2/a^2}\,\mathrm{d}x.$$

We then make the substitution $u = x/a$, so $x = au$ and $\mathrm{d}x = a\,\mathrm{d}u$. Also, $x = \pm\infty$ corresponds to $u = \pm\infty$, so

$$J = \frac{a^3}{b^2}\int_{-\infty}^{\infty} u^2\,\mathrm{e}^{-u^2}\,\mathrm{d}u = \frac{a^3}{b^2}\frac{\sqrt{\pi}}{2}.$$

Comments (1) You should never have to calculate a difficult integral from scratch in this course. If the integral looks difficult, ask yourself whether a change of variable will convert it into a known form or whether the integral is automatically equal to zero for reasons to be explored in the next exercise (Exercise 4.2).

(2) Note how the change of variable is accomplished. We first choose a new variable u that is a suitable function of x. The choice is made so that our integrand is converted into a multiple of the integrand that appears in a standard integral. We then convert the integrand, the limits of integration and the element of integration $\mathrm{d}x$ and use the standard integral.

Solution 4.2 Integrals I_2, I_3, I_7 and I_8 can immediately be recognized as being equal to zero because they are all of the form

$$\int_{-a}^{a} h(x)\,\mathrm{d}x,$$

where $h(x) = -h(-x)$ is an odd function integrated over a range that is centred on the origin. The remaining integrals, I_1, I_4, I_5 and I_6 are not of this form and are not necessarily equal to zero. (Depending on the functions involved, these integrals could be equal to zero in special cases, but no general conclusions can be drawn.)

In I_1, the integrand is an odd function, but the range of integration is not centred on the origin so we cannot claim that $I_1 = 0$. The remaining integrals are all over a range that is centred on the origin, so these integrals vanish if their integrands are odd functions.

The integrand in I_2 is odd because it is the product of an even function, $f(x)$, and an odd function, $g(x)$: even $\times$ odd $=$ odd. Similarly, the integrand in I_3 is odd because it is the product of an even function, $f^2(x)$, and an odd function, $g(x)$. The integrand in I_4 is not odd; it is the product of two even functions ($g^2(x)$ and $f(x)$) and so is itself even.

The remaining integrands involve derivatives. It is useful to note that the operator $\mathrm{d}/\mathrm{d}x$ can be regarded as an odd operator, playing a similar role to a multiplicative odd function. Explicitly, if $f(x)$ is an even function, then $h(x) = \mathrm{d}f(x)/\mathrm{d}x$ is an odd function because

$$h(-x) = \frac{\mathrm{d}f(-x)}{\mathrm{d}(-x)} = \frac{\mathrm{d}f(x)}{\mathrm{d}(-x)} = -\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -h(x).$$

The integrand in I_5 is not odd: it can be regarded as the product of an even function, $f(x)$, an odd operator, $\mathrm{d}/\mathrm{d}x$, and an odd function, $g(x)$: even × odd × odd = even. Similarly, the integrand in I_6 is not odd: it is the product of an odd function, $g(x)$, an odd operator, $\mathrm{d}/\mathrm{d}x$, and an even function, $f(x)$: odd × odd × even = even.

The integrand in I_7 is odd: it is the product of an even function, $f(x)$, an odd operator, $\mathrm{d}/\mathrm{d}x$, and an even function, $f(x)$: even × odd × even = odd. Similarly, the integrand in I_8 is odd: it is the product of an odd function, $g(x)$, an odd operator, $\mathrm{d}/\mathrm{d}x$, and an odd function, $g(x)$: odd × odd × odd = odd.

Comments (1) You can save a lot of time by using symmetry to spot definite integrals that are equal to zero. The principle used here is well worth remembering – the definite integral of any odd function over a range of that is centred on the origin is equal to zero. This method is further discussed in the solution to Exercise 4.6 of Book 1.

(2) When dealing with products of quantities, note that:

$$\begin{aligned} \text{even} \times \text{even} &= \text{even} \\ \text{odd} \times \text{odd} &= \text{even} \\ \text{even} \times \text{odd} &= \text{odd.} \end{aligned}$$

In this context, the first derivative operator $\mathrm{d}/\mathrm{d}x$ behaves as an odd operator. For example, if $f(x)$ is an even function, then $\mathrm{d}f/\mathrm{d}x$ is odd (formed from one odd thing and one even thing). Similarly, if $g(x)$ is an odd function, then $\mathrm{d}g/\mathrm{d}x$ is even (formed from two odd things).

Solution 4.3 The volume element in spherical coordinates is $r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$, so the required volume integral is

$$\begin{aligned} I &= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=R} r\sin\theta \times r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= 2\pi \int_0^{\pi} \sin^2\theta \, \mathrm{d}\theta \int_0^R r^3 \, \mathrm{d}r \\ &= 2\pi \times \frac{\pi}{2} \times \frac{R^4}{4} \\ &= \frac{\pi^2 R^4}{4}. \end{aligned}$$

Comment It is essential to use the appropriate volume element $r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$. The factors of r^2 and $\mathrm{d}r$ ensure that this has the dimensions of length^3, as expected for a volume. Note also that θ ranges from 0 to π, while ϕ ranges from 0 to 2π. To allow both angles to range from 0 to 2π would double-count the points on the surface of a sphere.

Topic 5 — Matrices

Solution 5.1 We have

$$\begin{aligned} \begin{bmatrix} 1 & -\mathrm{i} \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix} &= \begin{bmatrix} 1 & -\mathrm{i} \end{bmatrix} \begin{bmatrix} a + \mathrm{i}b \\ b + \mathrm{i}c \end{bmatrix} \\ &= (a + \mathrm{i}b) - \mathrm{i}(b + \mathrm{i}c) \\ &= a + c. \end{aligned}$$

Comment It is important to preserve to order of the matrices in a product such as ABC, but we can choose to group them either as $\mathrm{A(BC)}$ or as $\mathrm{(AB)C}$.

Solution 5.2 We have

$$AB = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 1 \times 2 & 3 \times 2 + 1 \times (-1) \\ 1 \times 1 + (-3) \times 2 & 1 \times 2 + (-3) \times (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \\ -5 & 5 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 \times 3 + 2 \times 1 & 1 \times 1 + 2 \times (-3) \\ 2 \times 3 + (-1) \times 1 & 2 \times 1 + (-1) \times (-3) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 \\ 5 & 5 \end{bmatrix},$$

so

$$AB - BA = \begin{bmatrix} 5 & 5 \\ -5 & 5 \end{bmatrix} - \begin{bmatrix} 5 & -5 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix}.$$

Since this is not equal to the zero matrix, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the matrices A and B do not commute with one another.

Comment Some matrices commute with each other, but most do not. The fact that matrices do not always commute has important consequences in quantum mechanics, especially for particles like electrons that have spin (Book 2 Chapter 2).

Solution 5.3 For the eigenvalue 9, the eigenvalue equation is

$$\begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 9 \begin{bmatrix} x \\ y \end{bmatrix}.$$

Multiplying out the matrices then gives

$$x + 4y = 9x$$

$$4x + 7y = 9y$$

so $y = 2x$ and a suitable normalized eigenvector for the eigenvalue 9 is

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where the factor $1/\sqrt{5}$ ensures that the eigenvector is normalized:

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5}(1^2 + 2^2) = 1.$$

For the eigenvalue -1, the eigenvalue equation is

$$\begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix}.$$

Multiplying out the matrices then gives

$$x + 4y = -x$$

$$4x + 7y = -y$$

so $x = -2y$ and a suitable normalized eigenvector for the eigenvalue -1 is

$$\frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

where the factor $1/\sqrt{5}$ ensures that the eigenvector is normalized:

$$\frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{5}((-2)^2 + 1^2) = 1.$$

The two eigenvectors are orthogonal because

$$\frac{1}{\sqrt{5}}\begin{bmatrix}-2 & 1\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\end{bmatrix} = \frac{1}{5}(-2+2) = 0.$$

Comment Your choices for the eigenvectors may be different from these: multiplying both components of one of our normalized eigenvectors by the same constant factor produces an equally valid eigenvector with the same eigenvalue. To preserve normalization, the constant factor would have to be of unit modulus (e.g. -1 or i, or more generally $\mathrm{e}^{\mathrm{i}}\alpha$ where α is real).

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